Affine Markov Chain Framework for Multi-firm Credit Migration

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Credit Spreads

- Recall credit spread on firm i at time t for maturity T is

$$\mathsf{CS}_t^i(T) = \frac{1}{T-t} \log\left(\frac{B_t^i(T)}{B_t(T)}\right)$$

- where $B_t^i(T)$ is price of default risky zero coupon bond,
- and $B_t(T)$ is price of riskless zero coupon bond.
- That is:

$$B_t^i(T) = B_t(T) e^{-(T-t)\mathsf{CS}_t^i(T)}$$

Simulated Credit Spread Dynamics for BB Firm

spreads

Basic Questions

Question

- *What are the principal drivers of credit spreads?*
- *Are existing models adequate to explain observed credit dynamics?*
- *Do these models adequately explain default correlations?*

Drivers of Credit Spreads

Collin-Dufresne, Goldstein, Martin 2001 found that the following factors are important drivers of CS_t^i:

- changes in firm's leverage

$$l_t^i := \frac{\text{Book value of debt}}{\text{Market value of equity} + \text{Book value of debt}}$$

- changes in spot rate r_t;
- returns on S&P500, as a proxy for changes in the economy; i.e. $s_t := \log \mathsf{SP}_t$.
- a single credit market specific factor "X" (they make a trial connection with the BBB credit spread index published monthly by Datastream).

Drivers of Credit Spreads (ctd)

Consider a regression for credit spreads on firm i:

$$\Delta \mathsf{CS}_t^i = \alpha^i + \beta_1^i \Delta l_t^i + \beta_2^i \Delta r_t + \beta_6^i \Delta s_t + \beta_7^i \Delta X_t + \epsilon_t^i$$

- This leads to $R^2 \sim 0.20$ if β_7^i is fixed 0
- and $R^2 \sim 0.60$ if β_7^i is freed.

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- Firm specific risks are predominantly coming from leverage alone;
- Market wide factors are more important than firm specific factors;
- Obvious market factors like term structure and stock indices account for about 20% of credit spread dynamics;
- A single unobserved factor X can account for about 40% of spread dynamics.

Questions

- What kind of mathematical models can match these observations?
- Can such a model be computable for $M = 100$ or more different firms?
- Can these systematic factors explain default correlation, in particular, implied correlations seen in CDO prices?
- Can we use such models to compute large-scale portfolio credit risk?

AMC Overview

Structural/intensity/credit migration/correlation pieces put together in a new and efficient way.

- TRH, Alexey Kuznetsov
    1. "Affine Markov chain models of multifirm credit migration", March 2005/September 2006
    2. "Fast CDO Computations in the Affine Markov Chain Model", March 2005/October 2006
- Related references: Jarrow-Lando-Turnbull 1997; Lando 1998; Arvanitis-Gregory-Laurent 1999; Duffie-Singleton 1999.

Assumptions

- We assume constant recovery fraction $w = 50\%$ (this for simplicity only);
- We have access to databases similar in scope to that of Collin-Dufresne, Goldstein, Martin 2001:
  1. time series of monthly closing treasury bond prices;
  2. SP500 return data;
  3. monthly closing corporate bond prices on hundreds of firms (eg 10 years of the Lehman Brothers bond database);
  4. time series of monthly leverage ratios or credit ratings on all firms (eg COMPUSTAT);
  5. An appropriate historical credit migration probability matrix;
  6. BBB credit spread index published monthly by Datastream, or similar.

AMC Ingredients I: Markov chains

- Three probability measures $P_0 \sim P \sim Q$:
  1. Reference "Thru-the-cycle" historical measure P_0 (credit migration dynamics is simple) ;
  2. Physical "Point-in-time" historical measure P (useful for interpretation and risk management, but forget it for now);
  3. Risk-neutral measure Q (complicated migration dynamics);
- Credit quality of firm i at time t $Y_t^i \in \{0, 1, 2, \ldots, K\}$
- Initial condition $Y_0^i = y^i$;
- Default time t^{i*}: first time Y_t^i hits 0.
- Y^i are time homogeneous Markov chains under P_0.

P_0 Migration Probabilities

- Long-term average migration rates

$$P_0[Y^i_{t+dt} = l | Y^i_t = k] = L_{kl} dt. \qquad (\text{for } k \neq l);$$

- Migration matrix L_{kl} with $L_{0l} = 0$.
- Migration probabilities starting from $Y^i_0 = y$:

$$P_0[1\{Y^i_t = l\}] = (e^{tL})_{yl} = \sum_{j=0}^{K} v_{yj} \tilde{v}_{jl} e^{-\alpha_j t}.$$

- $\tilde{V}^{-1} = V = (v_{kl})$ is eigenvector matrix of L, $-\alpha_j$ are (non-positive) eigenvalues.

Credit Ratings: Interpretation I

- Markov states represent Standard and Poor's "rating classes":

$$\{0,\ldots,7\} \leftrightarrow \{\text{"default"},\mathrm{CCC},\mathrm{B},\mathrm{BB},\mathrm{BBB},\mathrm{A},\mathrm{AA},\mathrm{AAA}\}$$

- Markov generator (historical):

$$L_Y = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.2856 & -0.4318 & 0.0928 & 0.0250 & 0.0142 & 0.0142 & 0.0000 & 0.0000 \\ 0.0753 & 0.0479 & -0.1928 & 0.0568 & 0.0073 & 0.0034 & 0.0021 & 0.0000 \\ 0.0273 & 0.0144 & 0.1181 & -0.2530 & 0.0813 & 0.0089 & 0.0025 & 0.0005 \\ 0.0049 & 0.0020 & 0.0174 & 0.0701 & -0.1711 & 0.0713 & 0.0047 & 0.0007 \\ 0.0010 & 0.0000 & 0.0048 & 0.0107 & 0.0688 & -0.1172 & 0.0309 & 0.0010 \\ 0.0000 & 0.0000 & 0.0030 & 0.0030 & 0.0105 & 0.0787 & -0.1043 & 0.0091 \\ 0.0000 & 0.0000 & 0.0000 & 0.0031 & 0.0020 & 0.0083 & 0.1019 & -0.1153 \end{pmatrix}$$

Leverage: Interpretation II

- Let Y_t^i represent firm's approximate leverage ratio;
- That is: for threshold values $0 = a_{K+1} < a_K < \cdots < a_1 = 1 < a_0 = \infty$,

$$\{Y_t^i = k\} = \{l_t^i \in [a_{k+1}, a_k)\}$$

- Then $L = (L_{kl})$ represents the generator of historical averaged migration rates from one leverage level to another.

More ingredients: Market factors

- Let $\mathbf{Z}_t = (r_t, s_t, X_t)$ (X may be higher dimensional).
- Under both P_0 and Q, SDEs for market factors $\mathbf{Z}_t$ do not depend on Y^i.
- Q migration dynamics:

$$Q[Y^i_{t+dt} = l | Y^i_t = k] = \lambda_t L_{kl} dt. \qquad (\text{for } k \neq l);$$

- Stochastic migration intensity or scalar credit risk premium:

$$\lambda_t = \beta_1 r_t + \beta_2 s_t + \beta_3 X_t$$

- NB: no firm specific risk factor is allowed here!!
- Stochastic time change $\tau_t = \int_0^t \lambda_s ds$.

Risk-neutral migration probabilities

- Let $\mathcal{G} \subset \mathcal{F}$ be filtration generated by $\mathbf{Z}$.
- Risk-neutral migration probabilities starting from $Y_0^i = y, \mathbf{Z}_0 = \mathbf{z}$:

$$
\begin{aligned}
E_{0,y,\mathbf{z}}[1\{Y_t^i = l\}] &= E_{0,\mathbf{z}}[E_{0,y}[1\{Y_t^i = l\}|\mathcal{G}]] = E_{0,\mathbf{z}}[(e^{L\int_0^t \lambda_u du})_{yl}] \\
&= \sum_{j=0}^{K} v_{yj}\tilde{v}_{jl} E_{0,\mathbf{z}}\left[e^{-\alpha_j \int_0^t \lambda_u du}\right].
\end{aligned}
$$

Affine Processes

- Assume $\mathbf{Z}_t = (r_t, s_t, X_t)$ is multivariate affine process
- This means there are computable functions $\Phi(t, \mathbf{u}, \mathbf{v}), \Psi(t, \mathbf{u}, \mathbf{v})$ s.t.

$$E_{0,\mathbf{z}}\left[e^{-\int_0^t \mathbf{u}\cdot\mathbf{Z}_s ds - \mathbf{v}\cdot\mathbf{Z}_t}\right] = e^{-\Phi(t,\mathbf{u},\mathbf{v})-\mathbf{z}\cdot\Psi(t,\mathbf{u},\mathbf{v})} := G(t,\mathbf{z};\mathbf{u},\mathbf{v}).$$

- Examples:

$$\begin{aligned} dr_t &= (a_1 - b_1 r_t) + \sqrt{2c_1 r_t} dW_t^1 \quad (\text{ CIR process}) \\ ds_t &= a_2 dt + c_2 dW_t^2 \\ dX_t &= -a_3 X_t dt + c_3 dJ_t \quad (\text{mean-reverting jump}) \end{aligned}$$

where J_t is increasing Levy jump process.

Result

Proposition

Defaultable zero coupon bond with fractional recovery w has price

$$B^{(d)}(T; \mathbf{Z}_0, y) = E_{0,\mathbf{Z}_0,y}\left[e^{-\int_0^T r_s ds}[1-(1-w)I\{t^* \leq T\}]\right]$$
$$= B(T) - (1-w)\sum_{j=0}^{K} v_{yj}\tilde{v}_{j0}G(T, \mathbf{Z}_0; (1+\alpha_j\beta_1, \alpha_j\beta_2, \alpha_j\beta_3), \mathbf{0})$$

Simulated Credit Spreads for All Firms

spreads

Credit Value-at-Risk Example

Portfolio of zero coupon bonds on M firms at time T:

$$X_T = \sum_{i=1}^{M} B^{(d)}(T^i - T, \mathbf{Z}_T, Y^i_T) = \sum_i G^i.$$

- $\mathsf{VaR}_{99\%}$ for time horizon $[0, T]$ is value a such that

$$\mathsf{Prob}(X_T < X_0 - a) = 1 - 0.99$$

- This is evaluation of "$m = 0$ tranche function" where

$$F^{(m)}(a) = E^P[((a - X_T)^+)^m], \quad m = 0, 1, \ldots$$

- Conditional value-at-risk or Tail VaR or expected shortfall:

$$\mathsf{CVaR}_{99\%} = F^{(1)}(X_0 - \mathsf{VaR}_{99\%})$$

- Expectations are with respect to historical measure P.

Approximating the tranche function

- Conditioned on $\mathbf{Z}_T$, $G^i|_{\mathbf{Z}_T}$ are independent RVs.
- Following Martin, Thompson, and Browne 2003, it's efficient to use saddlepoint method to compute conditional tranche functions

$$F^{(m)}(a|\mathbf{Z}_T) = E^P[((a - \sum_i G^i|_{\mathbf{Z}_T})^+)^m|\mathbf{Z}_T]$$

- Generate sufficiently dense set of values $\mathbf{Z}_T^1, \ldots, \mathbf{Z}_T^N$, either randomly (Monte Carlo) or on grid (numerical integration);
- For weights $\{w^n\}_{n=1}^N$, integrate over conditioning RVs;

$$F^{(m)}(a) \sim \sum_{n=1}^N w^n F^{(m)}(a|\mathbf{Z}_T^n)$$

- Total flop count: $\mathsf{const} \times K^2 \times M \times N$.

CDOs

- Collateralized Debt Obligations are contracts written on the cumulative loss $L(t)$ due to defaults in a portfolio of many credit default swaps or credit risky bonds.
- A large portfolio of similar bonds written on different firms is sliced into CDO tranches ordered by seniority.
- Each CDO tranche is a separate investment vehicle with its characteristic risk-reward.

Basic Ingredients for CDOs

- M reference credits "firms" with notional amounts of $N^j, j = 1, 2, \ldots, M$. These represent the face value of the individual bonds;
- Default time t^{j*} of firm j;
- Fractional recovery w^j (deterministic) after default of firm j;
- Loss l_j caused by default of firm j as fraction of total notional

$$l^j = (1 - w^j)N^j/N, \quad N = \sum_{j=1}^{M} N^j$$

- Cumulative portfolio loss $L(t)$ up to time t as fraction of total notional:

$$L(t) = \sum_j l^j I(t^{j*} \le t).$$

Synthetic CDO Tranche

- Credit swap between two parties, insured and insurer.
- Two components, premium leg and insurance leg are basic credit derivatives on total default loss $L(t)$ of portfolio.
- Fair price of premium leg with attachment level a:

$$V(a) = E^Q[\int_0^T e^{-rt}(a - L(t))^+ dt]$$

- Here tranche function is $F^{(1)}(a,t) = E^Q[(a - L(t))^+]$.

Synthetic CDO Tranche

- Fair price of insurance leg with attachment level a:

$$\begin{aligned} W(a) &= -E^Q\left[\int_0^T e^{-rt} d(a - L(t))^+\right] \\ &= a - e^{-rT} E^Q[(a - L(t))^+] - \int_0^T r e^{-rt} E^Q[(a - L(t))^+] d \end{aligned}$$

by Integration by Parts.

Synthetic CDO Tranche

- Fair price of insurance leg with attachment level a:

$$
\begin{aligned}
W(a) &= -E^Q\left[\int_0^T e^{-rt} d(a-L(t))^+\right] \\
&= a - e^{-rT}E^Q[(a-L(t))^+] - \int_0^T re^{-rt}E^Q[(a-L(t))^+]d
\end{aligned}
$$

  by Integration by Parts.
- Pays default losses within portfolio, up to level a.

Synthetic CDO Tranche

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$$= a - e^{-rT} E^Q[(a - L(t))^+] - \int_0^T re^{-rt} E^Q[(a - L(t))^+] d$$

by Integration by Parts.

- Pays default losses within portfolio, up to level a.
- $[a_-, a_+]$–tranche spread: that multiplier X of the premium tranche leg such that $X(V(a_+) - V(a_-)) = W(a_+) - W(a_-)$.

AMC Computation of CDOs

- For both legs, the critical computation boils down to

$$\int_0^T E^Q_{0,\mathbf{z},\mathbf{y}}[e^{-\int_0^t r_s ds}(a - L_t)^+]dt.$$

- Now $L_t|_{\mathbf{Z}} = L|_{\tau_t}$ is the loss distribution for M independent, time homogeneous Markov chains.
- Thus

$$E^Q_{0,\mathbf{z},\mathbf{y}}[e^{-\int_0^t r_s ds}(a - L_t)^+] = E^Q_{0,\mathbf{z}}[e^{-\int_0^t r_s ds} H(\tau_t, a)]$$

where $H(\tau_t, a) = E^Q_{0,\mathbf{y}}[(a - L_t)^+|\mathbf{Z}]$.
- A few more manipulations leads to...

Theorem on CDOs

Theorem (CDO Pricing)

- *Price of the premium leg (insurance leg is similar) with attachments levels* $[a_-, a_+]$*:*

$$V = \int_0^\infty [H(\tau, a_+) - H(\tau, a_-)] G^P(\tau, \mathbf{z}) d\tau$$

- $H(\tau, a) = E^Q[(a - L_t)^+ | \mathbf{Z}]$ *depends on Markov chain parameters, CDO composition, and initial credit ratings* $y_1, \ldots, y_M$.

-

$$G^P(\tau, \mathbf{z}) = \int_0^T E^Q_{0,\mathbf{z}}[e^{-\int_0^t r_s ds} \delta(\tau_t - \tau) dt]$$

depends only on market parameters.

Remarks

- Formula separates time change effects hidden in tranche–independent $G^P(\tau, \mathbf{z})$ from effects of conditional Markov chains $Y|_{\mathbf{Z}}$, conditional loss process $L|_{\mathbf{Z}}$ and attachment points a_-, a_+ hidden in tranche function $H(\tau, a)$.
- Extension to N_1 stochastic times $\vec{\tau} = (\tau_1, \ldots, \tau_{N_1})$:

$$V = \int_{(\mathbb{R}_+)^{N_1}} [H(\vec{\tau}, a_+) - H(\vec{\tau}, a_-)] G^P(\vec{\tau}, \mathbf{z}) d^{N_1}\vec{\tau}$$

- In general case $H(\tau, a) = E^Q[(a - L_t)^+ | \mathbf{Z}]$ can be accurately approximated by the Saddlepoint Method computed on an N_1 dimensional grid.

CDO Spreads (Normal approximation)

M	Tranche	Model A Lo Corr Normal	Model A Lo Corr Exact	Model B Hi Corr Normal	Model B Hi Corr Exact
20	[0,0.03]	1694.82	1918.01	1301.47	1423.47
	[0.03,0.07]	792.44	651.24	697.46	616.20
	[0.07,0.10]	277.14	281.46	381.23	378.46
	[0.10,0.15]	97.54	109.19	225.71	232.15
	[0.15,0.30]	11.53	12.57	74.03	74.46
	[0.30,1.00]	0.03	0.03	2.13	2.13

CDO Spreads (Normal approximation)

M	Tranche	Model A Lo Corr Normal	Model A Lo Corr Exact	Model B Hi Corr Normal	Model B Hi Corr Exact
100	[0,0.03]	2490.01	2480.87	1674.56	1670.82
	[0.03,0.07]	606.64	604.52	618.39	617.40
	[0.07,0.10]	190.07	191.04	353.50	353.68
	[0.10,0.15]	59.12	59.60	206.70	206.97
	[0.15,0.30]	4.99	5.02	63.22	63.23
	[0.30,1.00]	0.003	0.003	1.38	1.38

spread

Summary of Basic AMC Ingredients

- Migration matrix $L = (L_{kl})$, such that long-term economy-wide 1-year migration probabilities are

$$P_{kl} = (e^L)_{kl}$$

- Calibrated affine model for observable risk factors $Z_t = (r_t, s_t, X_t)$
- Calibrated migration intensity (speed of time change):

$$\lambda_t = \beta_1 r_t + \beta_2 s_t + \beta_3 X_t$$

Concluding Remarks

- AMC framework gives dynamical models of multifirm credit migration and default.
- AMC generalizes and unifies "intensity" and "structural" models.
- Dynamic correlation and credit spreads endogenous.
- Correlations arise through "stochastic time change".
- VaR and CDO computations are very efficient.

THE END

Final Question

How well do these ideas work in the real economy?