

# Tail asymptotics for a Lévy-driven tandem queue with an intermediate input and its extensions

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# Our plan

- 2 node tandem fluid queue
- What is our interest ?
- Modeling assumptions and basic facts
- Our approach
- The convergence domain of  $\varphi$
- Exact asymptotics by complex inversions
- Weak asymptotics for the Lévy input case
- Remarks on extensions
- References

# 2 node tandem fluid queue

$X_2(t)$  is called an intermediate input.

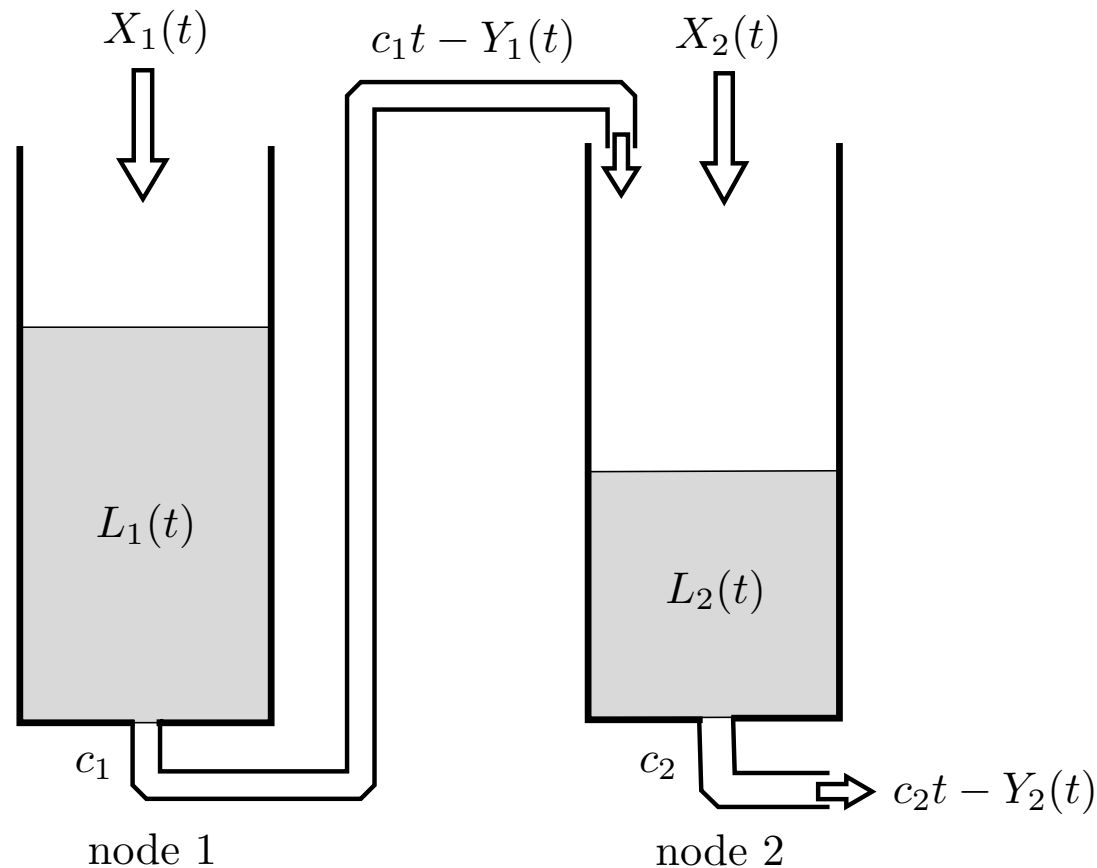


Figure 1: The buffer contents  $L_i(t)$ , accumulated inputs  $X_i(t)$  and outputs  $c_i t - Y_i(t)$  at time  $t$  for  $i = 1, 2$ .

# What is our interest ?

For the two node tandem fluid queue, we assume:

- The second node also has an exogenous input.
- The exogenous inputs are independent and subject to the **spectrally positive Lévy process**.
- Each node has a constant release rate.

We like to see how inputs influence its stationary distributions.

- No closed form result is available\*.
- Decay rates are known for the **Brownian inputs**.

⇒ Our interest is in **exact tail asymptotics** of the stationary distribution, particularly of the second buffer content.

\* This is different from the pure tandem queue.

# Background of this talk

This study is stimulated by

- the recent work of [Lieshout and Mandjes \(2008\)](#), which considers the pure tandem fluid queue.
- exact asymptotics on the double QBD of [Miyazawa \(2009\)](#)
- exact asymptotics on the random walk on a positive quadrant of [Foley and McDonald \(2005\)](#)

Those papers apply techniques which are **different from the large deviations**, particularly, sample path large deviations.

We are interested in the above approaches because

- exact asymptotics can be considered,
- they might be applicable for a higher dimensional case.

No explicit result is known for more than 2-node networks.

# About the Lévy process

A real valued process  $\{X(t)\}$  is said to be 1-dimensional Lévy process if it has independent and stationary increments, which can be decomposed into the following three components.

- $at + \sigma B(t)$ : Brownian motion with drift  $a$  and variance  $\sigma^2$ .
- $J^{(0)}(t) = \int_0^t \int_0^1 x(\Lambda(du, dx) - du\nu(dx))$  : Martingale with jumps not greater than 1.
- $J^{(1)}(t) = \int_0^t \int_{1+}^{\infty} x\Lambda(du, dx)$  : Compound Poisson process with jumps greater than 1,

where random measure  $\Lambda$  is a pure jump component of  $\{X(t)\}$ , and  $du\nu(dx) = E(\Lambda(du, dx))$ .  $\nu$  is called a spectral measure.

We assume that  $X(t)$  only has **positive jumps**. This may be reasonable for input processes.

# Lévy exponent

The moment generating function of Lévy process  $X(t)$  has the following form.

$$E(e^{\theta X(t)}) = e^{t\kappa(\theta)}, \quad \Re\theta \leq 0.$$

where  $\kappa(\cdot)$  is called an Lévy exponent, and given by

$$\kappa(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \kappa^{(0)}(\theta) + \kappa^{(1)}(\theta). \quad (1)$$

where

$$\kappa^{(0)}(\theta) = \int_0^1 (e^{\theta x} - 1 - \theta x)\nu(dx),$$

$$\kappa^{(1)}(\theta) = \int_{1+}^{\infty} (e^{\theta x} - 1)\nu(dx).$$

We put subscript  $i$  for characteristics for node  $i$ . For example,  $X_i(t)$ ,  $a_i$ ,  $\sigma_i$ ,  $\kappa_i(\theta)$ ,  $\kappa_i^{(0)}(\theta)$ ,  $\kappa_i^{(1)}(\theta)$ .

# Sample path of the tandem fluid queue

For node  $i$ , let  $X_i(t)$  be the accumulated input at time  $t \geq 0$ ,  $L_i(t)$  be the buffer content and  $c_i$  be a release rate, then

$$L_1(t) = L_1(0) + X_1(t) - c_1t + Y_1(t), \quad (2)$$

$$L_2(t) = L_2(0) + X_2(t) + c_1t - Y_1(t) - c_2t + Y_2(t), \quad (3)$$

where,  $Y_i(t)$  is a nondecreasing process which regulates  $L_i(t)$  to be nonnegative, and called a **regulator**. Then,  $(L_1(t), L_2(t))$  is called a reflecting process generated by

$(X_1(t) - c_1t, X_2(t) + c_1t - c_2t)$  through reflection matrix

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Since  $X_i(t)$  has no negative jump,  $Y_i(t)$  is continuous in  $t$ .

# Stationary equations

Let  $\lambda_i \equiv E(X_i(1))$ , and assume the stability condition:

$$\lambda_1 < c_1, \quad \lambda_1 + \lambda_2 < c_2,$$

then the stationary distribution exists. Denote it by  $\nu$ . Let

$\mathbf{L} \equiv (L_1, L_2)$  be a random vector subject to  $\nu$ , and define

$$\varphi(\theta_1, \theta_2) = E(e^{\theta_1 L_1 + \theta_2 L_2}), \quad \varphi_i(\theta_j) = E_\nu \left( \int_0^1 e^{\theta_j L_j(t)} Y_i(dt) \right).$$

where  $E_\nu$  stands for the conditional expectation given  $\mathbf{L}(0)$

having  $\nu$ . Applying **Itô's formula**\* to (2) and (3), we have

**Proposition 1** For  $\theta_1, \theta_2 \leq 0$ ,

$$\gamma(\theta_1, \theta_2) \varphi(\theta_1, \theta_2) = (\theta_1 - \theta_2) \varphi_1(\theta_2) + \theta_2 \varphi_2(\theta_1), \quad (4)$$

where,  $\gamma(\theta_1, \theta_2) = c_1 \theta_1 + (c_2 - c_1) \theta_2 - \kappa_1(\theta_1) - \kappa_2(\theta_2)$ .

\* We can also use **Kella-Whitt martingale** for this derivation.

# Rough and exact asymptotics

Since stationary distribution  $\nu$  is two dimensional, there may be many ways to consider its asymptotics. We consider the tails of  $L_1, L_2, d_1L_1 + d_2L_2$ , where  $d_i$ 's are positive constants.

We define their asymptotics in the following way. For positive function  $g(x)$  for  $x \in [0, \infty)$ ,

$$\bar{\alpha} = -\limsup_{x \rightarrow \infty} \frac{1}{x} \log g(x), \quad \underline{\alpha} = -\liminf_{x \rightarrow \infty} \frac{1}{x} \log g(x)$$

are called **upper and lower decay rates** of  $g(x)$ . If  $\alpha = \bar{\alpha} = \underline{\alpha}$ , then  $\alpha$  is called a **decay rate**. If there is a positive function  $h$  such that

$$\lim_{x \rightarrow \infty} \frac{g(x)}{h(x)} = 1,$$

then  $h(x)$  is said to be **exact asymptotics** of  $g(x)$ , and write it as  $g(x) \sim h(x)$ .

# What is known around this model

- Brownian networks (BN) and Lévy networks (LN)
  - Reflection map for BN – Harrison and Reiman (1981)
  - LST for tandem LN – Kella and Whitt (1992)
  - LST for out-tree LN – Dębicki, Dieker and Rolski (2007)
  - Sample path LD – Majewski (1998)
  - Rough asymptotics for 2 node BN
    - Avram, Dai and Hasenbein (2001)
- Discrete-time queueing networks
  - Rough asymptotics for intree networks – Chang (1995)
  - Rough and Exact asymptotics for 2 node networks
    - Borovkov and Mogul'skii (2001)

Already mentioned about Lieshout and Mandjes (2008), Foley and McDonald (2005) and Miyazawa (2009).

# Our approach

Tail asymptotics of the stationary distributions of queueing networks have been studied by

- Sample path large deviations ([\[1, 3\]](#))
- Spectra or convergence radius ([\[5\]](#)).
- Markov renewal theorem.
- Complex analysis and/or local limit theorems.
- Tauberian theorems when LST is known ([\[4\]](#)).

Our approach of [\[6\]](#) is close to the last one, but

1. we first find singular points through the **convergence domain** of the joint moment generating function  $\varphi$ ,
2. then apply **complex inversions** for asymptotics due to [\[2\]](#).

# Stationary distribution of node 1

Let  $\theta_2 = 0$  in (4), then

$$(c_1\theta_1 - \kappa_1(\theta_1))\varphi(\theta_1, 0) = \theta_1\varphi_1(0).$$

Thus, we have **Pollaczek-Khinchine formula**:

$$\varphi(\theta_1, 0) = \frac{(c_1 - \lambda_1)\theta_1}{(c_1\theta_1 - \kappa_1(\theta_1))}. \quad (5)$$

From this formula, if there exists  $\alpha_1 > 0$  satisfying

$$c_1\alpha_1 = \kappa_1(\alpha_1),$$

then  $\alpha_1$  is the rough decay rate of  $P(L_1 > x)$ . Furthermore, if, for some  $\epsilon > 0$ ,  $k_1(\theta)$  is finite for  $\theta < \alpha_1 + \epsilon$ , then

$$P(L_1 > x) \sim Ce^{-\alpha_1 x}$$

for some constant  $C > 0$  since  $z = \alpha_1$  is a simple pole of the analytic function  $\varphi(z, 0)$ .

# Brownian input case: the domain of $\varphi$

For simplicity, assume that  $X_i(t)$  is continuous in time  $t$ , that is, a Brownian motion. Let  $r_1 = c_1 - \lambda_1$ ,  $r_2 = c_2 - c_1 - \lambda_2$ , then

$$\gamma(\theta_1, \theta_2) = r_1\theta_1 + r_2\theta_2 - \frac{1}{2}(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2),$$

and therefore stationary equation (4) becomes

$$\begin{aligned} (r_1z_1 + r_2z_2 - \frac{1}{2}\sigma_1^2(z_1^2 + \sigma_2^2z_2^2))\varphi(z_1, z_2) \\ = (z_1 - z_2)\varphi_1(z_2) + z_2\varphi_2(z_1). \end{aligned} \quad (6)$$

Thus, we have the following facts.

**Lemma 1** The left side of (6) is **an analytic function of two variables**  $\iff \varphi_1(z)$  and  $\varphi_2(z)$  are **analytic functions of  $z$** .

Remark:  $f(z_1, z_2)$  is said to be an analytic function of two variable if  $f(z_1, z_2)$  is analytic for each  $z_i$ .

# Further auxiliary lemmas

From the stationary equation (6),

$$\varphi_1(\theta_2) = \left(\frac{1}{2}\sigma_2^2\theta_2 - r_2\right)\varphi(0, \theta_2) + r_1 + r_2.$$

Since

$$\varphi(0, \theta_2) = \frac{\varphi_1(\theta_2) - \varphi_1(z_2)}{\frac{1}{2}\sigma_2^2\theta_2 - r_2},$$

$z_2 = \frac{2r_2}{\sigma_2^2}$  is a removable singular point of  $\varphi(0, z)$ . Hence,

**Lemma 2**  $\varphi_1(z)$  and  $\varphi(0, z)$  have the same singularity.

**Lemma 3**  $\varphi_2(\theta)$  is finite for  $\theta < \alpha_1$ , where  $\alpha_1 = \frac{2r_1}{\sigma_1^2}$ .

**Remark 1** The domain of  $\varphi_2(\theta)$  may be larger than  $\theta < \alpha_1$ .

# Ideas for finding the domain

Let  $\mathcal{D} = \{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi(\theta_1, \theta_2) < \infty\}$ , which is the convergence domain of  $\varphi$ . How we can get it from stationary equation (6).

- **Boundary free kernel** : Let  $\Gamma_0 = \{\boldsymbol{\theta} \in \mathbb{R}^2; \gamma(\boldsymbol{\theta}) = 0\}$ .  
Since  $\gamma(\boldsymbol{\theta}) = r_1\theta_1 + r_2\theta_2 - \frac{1}{2}(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2)$ ,  $\Gamma_0$  is ellipse.  
Denote its inside by  $\Gamma_+$  and outside by  $\Gamma_-$ .
- **Signs** : Since  $\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) + (\theta_2 - \theta_1)\varphi_1(\theta_2) = \theta_2\varphi_2(\theta_1)$ , if  $\boldsymbol{\theta} \in \Gamma_+$  and  $\theta_2 > \theta_1$ , then  $\varphi_2(\theta_1)$  is finite if and only if  $\varphi(\boldsymbol{\theta})$  and  $\varphi_1(\theta_2)$  are finite, that is,  $\boldsymbol{\theta} \in \mathcal{D}$ . Furthermore,  
 $\theta_1 < \alpha_1 \Leftrightarrow \varphi(\theta_1, 0) < \infty \Rightarrow \varphi_2(\theta_1) < \infty$ . Hence,  
 $\boldsymbol{\theta} \in \Gamma_+, \theta_2 > \theta_1, \theta_1 < \alpha_1 \Rightarrow \boldsymbol{\theta} \in \mathcal{D}$ .
- **Check all combinations** :  $\Gamma_0^+$  or  $\Gamma_0^-$  and the signs of  $\theta_2 - \theta_1$  and  $\theta_1$ ,  $2^3 = 8$  cases in total are examined.

# Notation for the convergence domain

Since  $\gamma(\theta_1, \theta_2) = 0$  is ellipse, we can define:

$$(\theta_1^{\max}, \theta_2^{\max}) = \arg \max_{(\theta_1, \theta_2)} \{\theta_2; \gamma(\theta_1, \theta_2) = 0\},$$

$$(\theta_1^{\min}, \theta_2^{\min}) = \arg \min_{(\theta_1, \theta_2)} \{\theta_2; \gamma(\theta_1, \theta_2) = 0\},$$

$$(\eta_1^{\max}, \eta_2^{\max}) = \arg \max_{(\theta_1, \theta_2)} \{\theta_1; \gamma(\theta_1, \theta_2) = 0\},$$

$$(\eta_1^{\min}, \eta_2^{\min}) = \arg \min_{(\theta_1, \theta_2)} \{\theta_1; \gamma(\theta_1, \theta_2) = 0\},$$

$$\beta = \max\{\theta; \gamma(\theta, \theta) = 0\} = \frac{2(r_1 + r_2)}{\sigma_1^2 + \sigma_2^2},$$

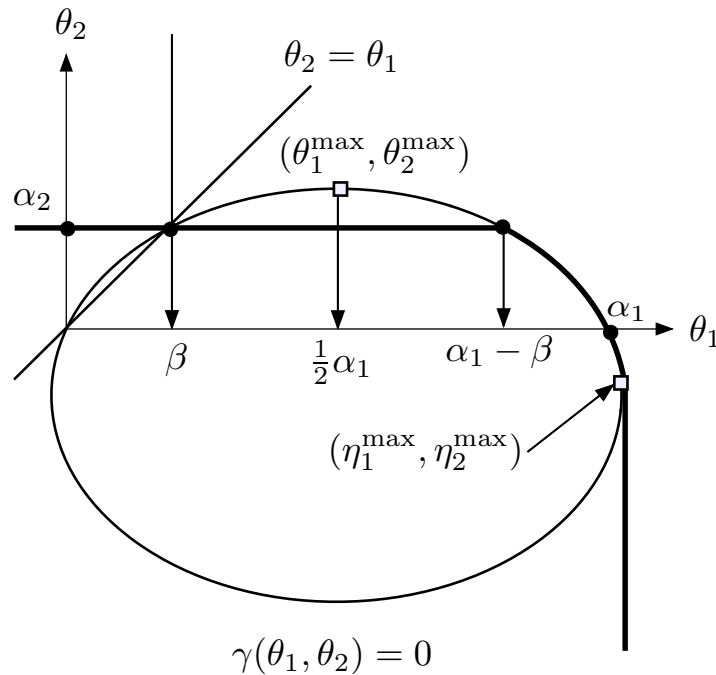
$$\xi_1(\theta_2) = \min\{\theta_1; \gamma(\theta_1, \theta_2) = 1\},$$

$$\xi_2(\theta_1) = \max\{\theta_2; \gamma(\theta_1, \theta_2) = 1\}.$$

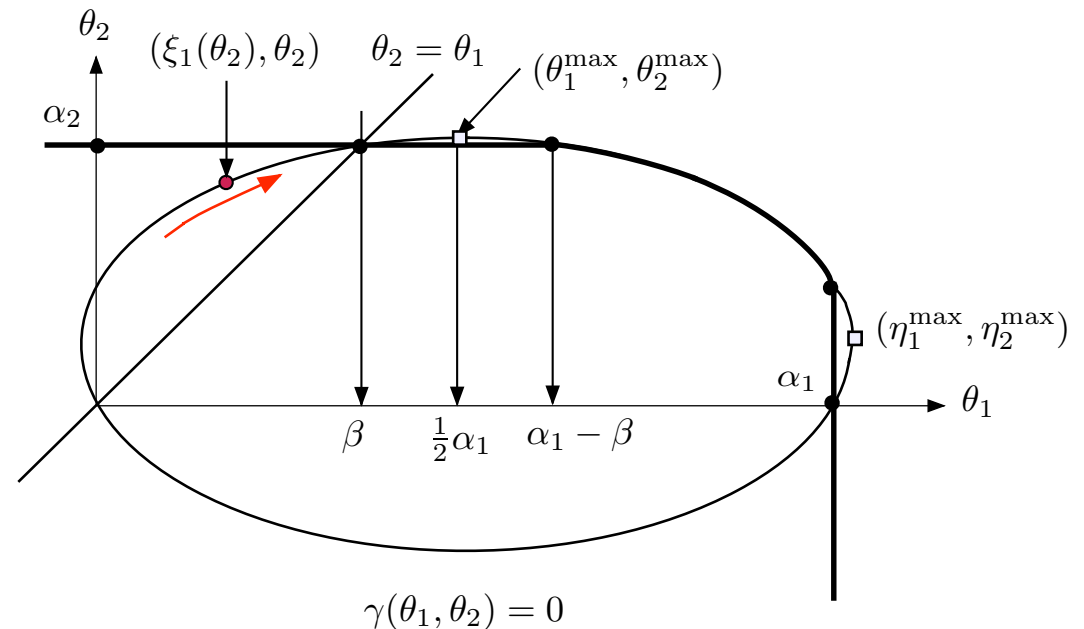
From the stability condition  $r_1 > 0, r_1 + r_2 > 0$ , we have  $\beta > 0$ .

# Shape of the convergence domain 1

Denote the interior of  $\mathcal{D} = \{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi(\boldsymbol{\theta}) < \infty\}$  by  $\mathcal{D}^\circ$ . The following figures are for the case that  $\beta < \frac{1}{2}\alpha_1$ .



(a-)  $\beta < \frac{1}{2}\alpha_1$  with  $r_2 < 0$

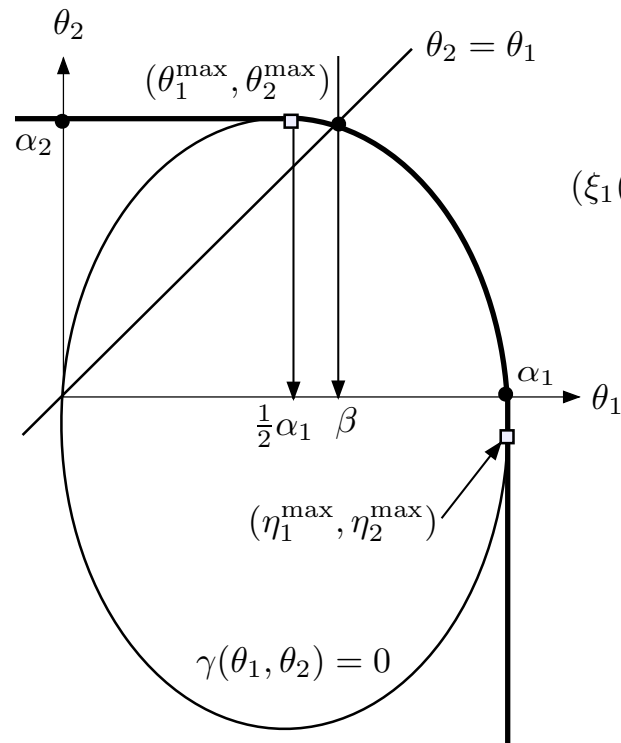


(a+)  $\beta < \frac{1}{2}\alpha_1$  with  $r_2 > 0$

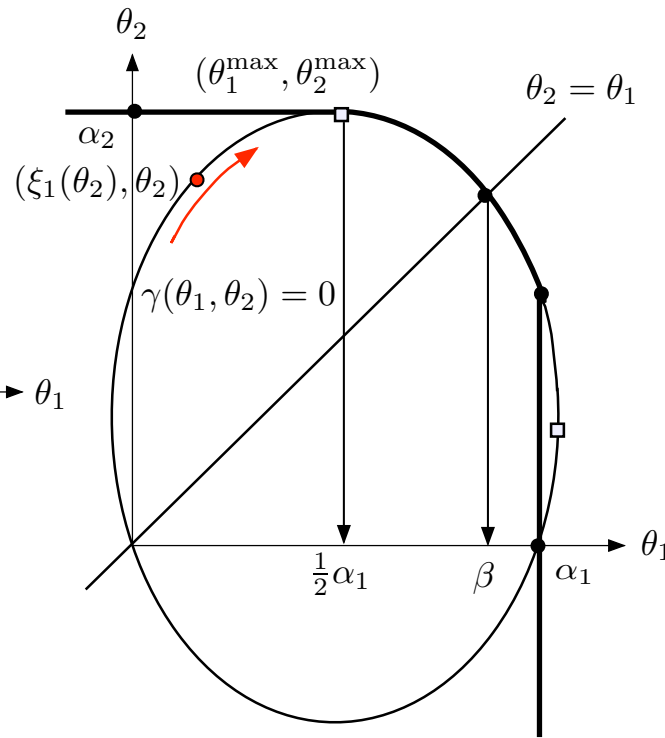
The domain  $\mathcal{D}^\circ$  is under the bold face curves.

# Shape of the convergence domain 2

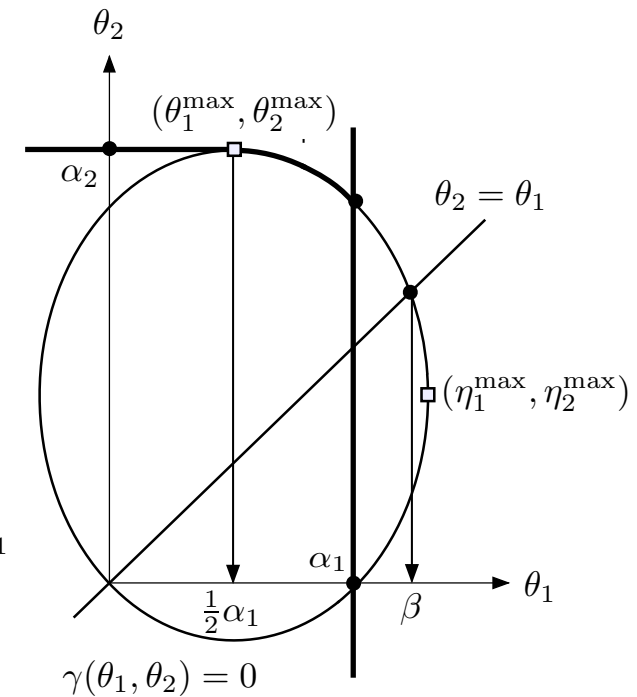
The following figures are for the case that  $\beta > \frac{1}{2}\alpha_1$ .



(b-)  $\frac{1}{2}\alpha_1 \leq \beta \leq \alpha_1$  with  $r_2 < 0$



(b+)  $\frac{1}{2}\alpha_1 \leq \beta \leq \alpha_1$  with  $r_2 > 0$



(c)  $\alpha_1 < \beta$

The domain  $\mathcal{D}^\circ$  is under the bold face curves.

# Answer to the convergence domain

Define  $\alpha_1^{\max}$  as

$$\alpha_1^{\max} = \begin{cases} \alpha_1, & \eta_2^{\max} \geq 0, \\ \eta_1^{\max}, & \eta_2^{\max} < 0, \end{cases}$$

Define the following two sets:

$$\mathcal{D}_1 = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1^{\max}, \theta_2 < \beta_2^{\max}\},$$

$$\mathcal{D}_2 = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \theta'_1, \theta_2 < \theta'_2,$$

for some  $(\theta'_1, \theta'_2)$  such that  $\gamma(\theta'_1, \theta'_2) \geq 0\}$ .

Then,

**Proposition 2** If  $\{X_i(t)\}$ 's are Brownian motions, then the interior of convergence domain  $\mathcal{D}^\circ = \mathcal{D}_1 \cap \mathcal{D}_2$  (see Figure 1).

**Remark 2** This result is **straightforwardly extended to the Lévy input case** if  $\kappa(\theta)$  is finite on a sufficiently large set.

# Asymptotics for the simple pole case

Let  $\psi(\theta) = \int_0^\infty e^{\theta x} \overline{F}(x) dx$  for the tail of distribution  $\overline{F}(x)$ .

(S1) If there are  $\alpha, p, q$  satisfying  $p < \alpha < q$  and integer  $k \geq 1$  such that

(S1a)  $\psi(z)$  is analytic for  $p \leq \Re z \leq q$  except for  $z = \alpha$ ,

(S1b)  $\psi(z)$  uniformly converges to 0 as  $z \rightarrow \infty$  for  $p \leq \Re z \leq q$ , and the integral  $\frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-ixy} \psi(q + iy) dy$  uniformly converges for  $x > T$ ,

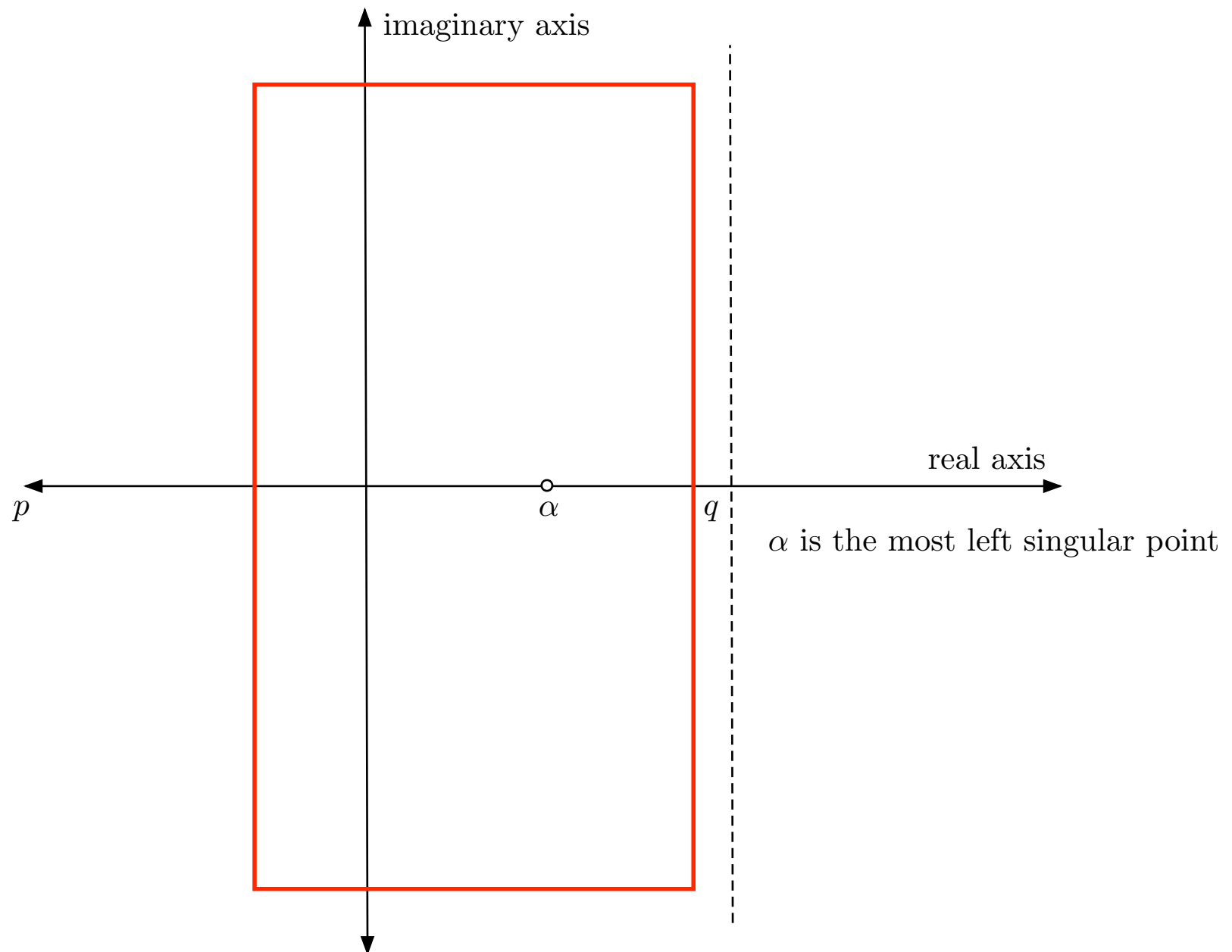
(S1c)  $\lim_{z \rightarrow \alpha} (\alpha - z)^k \psi(z) = C'_1$  for some  $C'_1 > 0$ ,

then

$$\overline{F}(x) = \frac{C'_1}{\Gamma(k)} x^{k-1} e^{-\alpha x} (1 + o(1)),$$

where  $\Gamma(z)$  is the gamma function.

# Counter integral for the simple pole



# Asymptotics for the branch point case

(S2) If there are  $\alpha > 0$  and  $\delta \in [0, \frac{\pi}{2})$  such that

(S2a)  $\psi(z)$  is analytic in the region:

$$\mathcal{G}_\alpha(\delta) \equiv \{z \in \mathbb{C}; \Re z > 0, z \neq \alpha, |\arg(z - \alpha)| > \delta\},$$

where  $\arg z$  is the principal part of the argument of complex number  $z$ ,

(S2b)  $\psi(z) \rightarrow 0$  as  $|z| \rightarrow \infty$  for  $z \in \mathcal{G}_\alpha(\delta)$ ,

(S2c) for some constant  $K$  and non integer real number  $s$

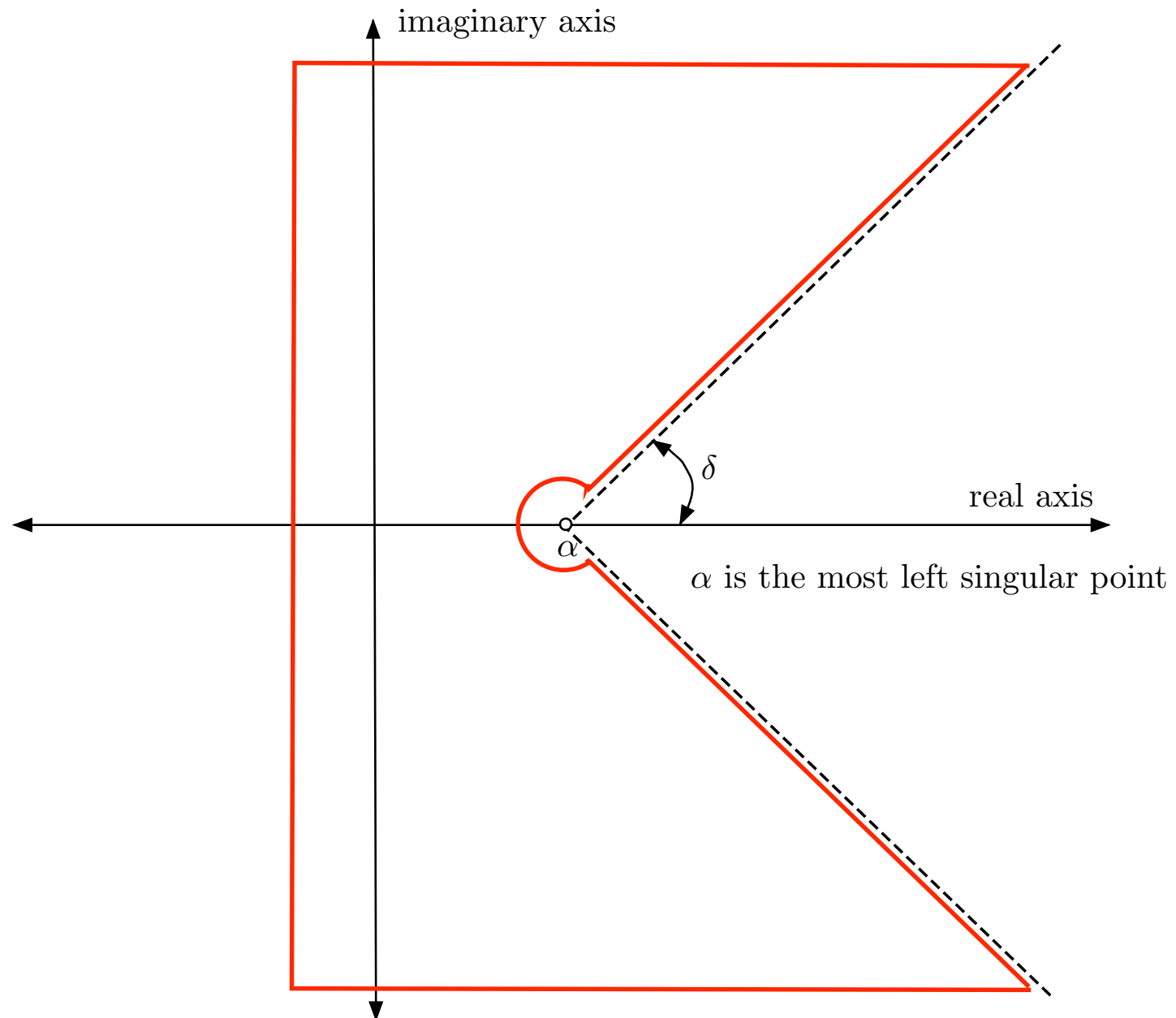
$$\psi(z) = K \mathbf{1}(s > 0) - C'_2(\alpha - z)^s + o((\alpha - z)^s),$$

for  $\mathcal{G}_\alpha(\delta) \ni z \rightarrow \alpha$ ,

where  $K$  must be  $\psi(\alpha)$  if  $s > 0$ , then

$$\bar{F}(x) = \frac{C'_2}{\Gamma(-s)} x^{-s-1} e^{-\alpha x} (1 + o(1)).$$

# Counter integral for the branch point



# The Brownian input case

**Theorem 1**  $P(L_2 > x)$  has the following exact asymptotics.

$$(1a) \quad \beta < \frac{1}{2}\alpha_1 \Rightarrow h(x) = C_1 e^{-\beta x}.$$

$$(2b) \quad \frac{1}{2}\alpha_1 = \beta \Rightarrow h(x) = C_2 x^{-\frac{1}{2}} e^{-\theta_2^{\max} x}.$$

$$(3c) \quad \frac{1}{2}\alpha_1 < \beta \Rightarrow h(x) = C_3 x^{-\frac{3}{2}} e^{-\theta_2^{\max} x}.$$

where  $C_1, C_2, C_3$  are positive constants obtained in terms of  $\varphi_2(\theta)$ , and

$$\alpha_1 = \frac{2r_1}{\sigma_1^2}, \quad \beta = \frac{2(r_1 + r_2)}{\sigma_1^2 + \sigma_2^2}, \quad \theta_2^{\max} = \frac{1}{\sigma_2^2} \left( r_2 + \sqrt{r_2^2 + r_1^2 \frac{\sigma_2^2}{\sigma_1^2}} \right).$$

**Remark 3** These three cases corresponds with those of a two-dimensional skip free and reflected random walk on the nonnegative integer quadrant (called a double QBD in [5]), but  $h(x) = Cx e^{-\alpha_1 x}$  **occurs in the latter case.**

# The Lévy input case

**Proposition 3** If  $\gamma(\theta, \theta) = 0$  has a positive solution, denoted by  $\beta$ , in addition to some finiteness conditions on  $\gamma$ , then, letting  $\theta_1^{\max} = \xi_1(\theta_2^{\max})$ , we have

(2a) If  $\beta < \theta_1^{\max}$ , then  $\int_0^x e^{\beta u} P(L_2 > u) du \sim C_1 x$ .

(2b) If  $\beta = \theta_1^{\max}$ , then  $\int_0^x e^{\theta_2^{\max} u} P(L_2 > u) du \sim 2C_2 x^{\frac{1}{2}}$ .

(2c) If  $\beta > \theta_1^{\max}$ , then  $\int_x^\infty e^{\theta_2^{\max} u} P(L_2 > u) du \sim 2C_3 x^{-\frac{1}{2}}$ .

Here,  $f(x) \sim g(x)$  if  $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ , and

$$C_1 = \frac{\beta \varphi_2(\beta)}{(\xi_1'(\beta) - 1)(\tilde{\kappa}_2(\beta) - r_2 \beta)} \text{ and } C_2 = \frac{\theta_2^{\max} \varphi_2(\theta_1^{\max})}{\tilde{\kappa}_2(\theta_2^{\max}) - r_2 \theta_2^{\max}} \sqrt{\frac{-\xi_2''(\theta_1^{\max})}{2\pi}},$$

where  $\xi_1'(\beta) = \frac{\tilde{\kappa}_2'(\beta) - r_2}{r_1 - \tilde{\kappa}_1'(\beta)}$  and  $\xi_2''(\theta_1^{\max}) = \frac{\tilde{\kappa}_1''(\theta_1^{\max})}{r_2 - \tilde{\kappa}_2'(\theta_2^{\max})}$ , and

$$C_3 = \frac{\theta_2^{\max} (\varphi_2(\theta_1^{\max}) + (\theta_2^{\max} - \theta_1^{\max}) \varphi_2'(\theta_1^{\max}))}{(\tilde{\kappa}_2(\theta_2^{\max}) - r_2 \theta_2^{\max}) (\theta_2^{\max} - \theta_1^{\max})^2 \sqrt{-2\pi \xi_2''(\theta_1^{\max})}}.$$

# Conjectures on the Lévy input case

**Conjecture 1**  $P(L_2 > x)$  has the following exact asymptotics.

$$(3a) \quad \beta < \frac{1}{2}\alpha_1 \Rightarrow h(x) = C_1 e^{-\beta x}.$$

$$(3b) \quad \frac{1}{2}\alpha_1 = \beta \Rightarrow h(x) = C_2 x^{-\frac{1}{2}} e^{-\theta_2^{\max} x}.$$

$$(3c) \quad \frac{1}{2}\alpha_1 < \beta \Rightarrow h(x) = C_3 x^{-\frac{3}{2}} e^{-\theta_2^{\max} x}.$$

where  $C_1, C_2, C_3$  are positive constants.

Remark: If  $P(L_2 > x) \sim x^u e^{-vx}$  for some  $u, v$ , then Proposition 3 proves this conjecture.

Why is it difficult to prove the conjecture ?

- To apply the complex inversion, we need to find the function  $\xi_1(z)$  which is implicitly determined by

$$\gamma(\xi_1(z), z) = 1$$

and analytic in some left-half plane.

# Remarks on extensions

The present formulation can be extended to a  $n$ -node network with a  $n$ -dimensional Lévy input and reflection matrix  $R$  such that

$$\mathbf{L}(t) = \mathbf{L}(0) + \mathbf{X}(t) - tR\mathbf{c} + R\mathbf{Y}(t),$$

where  $R = I - P^T$  for routing matrix  $P$ ,  $\mathbf{X}(t)$  is the Lévy process,  $\mathbf{L}(t)$  is a buffer content and  $\mathbf{Y}(t)$  is a regulator. Under the stability condition  $(I - P)^{-1}E(\mathbf{X}(1)) < \mathbf{c}$ , the mgf  $\varphi$  of the stationary distribution of  $\mathbf{L}(t)$  satisfies

$$\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) = \langle \boldsymbol{\theta}, R\underline{\varphi}(\boldsymbol{\theta}) \rangle, \quad (7)$$

where  $\gamma(\boldsymbol{\theta}) = \langle \boldsymbol{\theta}, R\mathbf{c} \rangle - \kappa(\boldsymbol{\theta})$  with Lévy exponent  $\kappa(\boldsymbol{\theta})$ , and

$$\underline{\varphi}(\boldsymbol{\theta}) = (\varphi_1(\boldsymbol{\theta}_1[0]), \dots, \varphi_n(\boldsymbol{\theta}_n[0]))^T,$$

where  $\varphi_i(\boldsymbol{\theta}_i[0]) = E_\pi \int_0^1 e^{\langle \boldsymbol{\theta}_i[0], \mathbf{L}(u) \rangle} dY_i(u)$ .

# How to get the domain of $\varphi$ ?

Let us consider the case that  $n = 3$ . In this case,

$$\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) = \gamma_1(\boldsymbol{\theta})\varphi_1(\theta_2, \theta_3) + \gamma_2(\boldsymbol{\theta})\varphi_2(\theta_1, \theta_3) + \gamma_3(\boldsymbol{\theta})\varphi_3(\theta_1, \theta_2),$$

where

$$\gamma_i(\boldsymbol{\theta}) = \theta_i - (p_{i1}\theta_1 + p_{i2}\theta_2 + p_{i3}\theta_3), \quad i = 1, 2, 3.$$

Letting  $\theta_1 = \theta_2 = 0$  and rearranging the terms,

$$\gamma(\boldsymbol{\theta})\varphi(\theta_1, 0, 0) - \gamma_2(\boldsymbol{\theta})\varphi_2(\theta_1, 0) - \gamma_3(\boldsymbol{\theta})\varphi_3(\theta_1, 0) = \gamma_1(\boldsymbol{\theta})\varphi_1(0, 0),$$

Hence, by similar arguments, for  $\boldsymbol{\theta} > \mathbf{0}$ ,

$$\gamma(\theta_1, 0, 0) > 0 \Rightarrow \varphi(\theta_1, 0, 0), \varphi_2(\theta_2, 0), \varphi_3(\theta_1, 0) < \infty.$$

$$\gamma(0, \theta_2, 0) > 0 \Rightarrow \varphi(0, \theta_2, 0), \varphi_1(\theta_2, 0), \varphi_3(0, \theta_3) < \infty.$$

$$\gamma(0, 0, \theta_3) > 0 \Rightarrow \varphi(0, 0, \theta_3), \varphi_1(0, \theta_3), \varphi_2(0, \theta_3) < \infty.$$

These are building blocks to find the convergence domain of  $\varphi(\boldsymbol{\theta})$ .

# Problems

- Exact asymptotics for the 2-node tandem fluid queue driven by the Lévy input.
- Exact asymptotics in all directions for the 2-dimensional Brownian and Lévy networks
- Rough asymptotics for the  $n$ -dimensional Brownian and Lévy networks.

One can reasonably conjecture the solutions for these problems.

However, their verification would require

- local limit theorems along the boundary faces,
- interference across boundary faces,
- lower bounds for logarithmic tail probabilities.

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