Equity Correlation Swaps: A New Approach For Modelling & Pricing

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“Finance and Derivatives teaches all of the fundamentals of quantitative finance clearly and concisely without going into unnecessary technicalities. You’ll pick up the most important theoretical concepts, tools and vocabulary without getting bogged down in arcane derivations or enigmatic theoretical considerations.”

– Paul Wilmott
Agenda

1. Fundamentals of index variance, constituent variance and correlation
2. Toy model for derivatives on realised variance
3. Rational pricing of correlation swaps
1. Fundamentals of index variance, constituent variance and correlation
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1.1 Realised and Implied Correlation
1.2 Correlation Proxy
1.3 Application: Variance Dispersion Trading
Realised and Implied Correlation

- **Realised Correlation**
  - **Pair of stocks**: statistical coefficient of correlation between the two time series of daily log-returns
  - **Basket of N stocks**: average of the N(N-1)/2 pair-wise correlation coefficients

- **Implied Correlation**
  - **Pair of stocks**: usually unobservable
  - **Basket of N stocks**: occasionally observable through quotes on basket calls or puts from exotic desks
  - **Liquid indices**: observable for listed strikes and maturities
Realised and Implied Correlation

- **Realised Correlation Definitions** (Equal Weights Assumption)
  - **Average pair-wise** (‘naive’) definition:
    \[
    \rho_{\text{Pairwise}} \equiv \frac{2}{N(N-1)} \sum_{i<j} \rho_{i,j}
    \]
  - **Canonical** (econometric) definition:
    \[
    \rho_{\text{Canonical}} \equiv \frac{\sigma^2_{\text{Index}} - \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2}{\sqrt{\frac{1}{N} \sum_{i=1}^{N} \sigma_i^2} \sqrt{\frac{1}{N} \sum_{i=1}^{N} \sigma_i^2}} \approx \frac{\sum_{i<j} \sigma_i \sigma_j \rho_{i,j}}{\sum_{i<j} \sigma_i \sigma_j}
    \]
Realised and Implied Correlation

- **Implied Correlation Definition** (Equal Weights Assumption)
  - No ‘naive’ definition (pair-wise implied correlations not observable)
  - Canonical (econometric) definition:
    \[
    \rho_{\text{Canonical}}^* \equiv \frac{\sigma_{\text{Index}}^* - \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^*}{\sqrt{\frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^*} - \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^*}
    \]
  - Note that the implied volatility surface translates into an implied correlation surface. We use fair variance swap strikes for $\sigma^*$’s unless mentioned otherwise.
1. Fundamentals of index variance, constituent variance and correlation

1.1 Realised and Implied Correlation

1.2 Correlation Proxy

1.3 Application: Variance Dispersion Trading
Correlation Proxy

- The previous definitions are easily generalised to arbitrary index weights

- **Proxy Formula**: Under certain regularity conditions on the weights, residual volatility becomes negligible and we have:

\[
\begin{align*}
\rho_{\text{Canonical}} \quad &\xrightarrow{N \to +\infty} \quad \left( \frac{\sigma_{\text{Index}}}{\bar{\sigma}_{\text{Constituent}}} \right)^2 \equiv \hat{\rho} \\
\rho^*_{\text{Canonical}} \quad &\xrightarrow{N \to +\infty} \quad \left( \frac{\sigma^*_{\text{Index}}}{\bar{\sigma}^*_{\text{Constituent}}} \right)^2 \equiv \hat{\rho}^*
\end{align*}
\]

- **Condition**: \( \frac{\text{MaxWeight}}{\text{MinWeight}} = o\left(\sqrt{N}\right) \)
Correlation Proxy

- Correlation (realised and implied) is thus close to the ratio of index variance to average constituent variance

\[
\text{Correlation} \approx \frac{\text{Index Variance}}{\text{Average Constituent Variance}}
\]

- This is interesting because index variance and average constituent variance can be traded on the OTC variance swap market
1. Fundamentals of index variance, constituent variance and correlation

1.1 Realised and Implied Correlation

1.2 Correlation Proxy

1.3 Application: Variance Dispersion Trading
Application: Variance Dispersion Trading

- **Variance Dispersion Trades**
  Spread of variance swap positions between an index and its constituents, usually:
  - **Long Average Constituent Variance**
  - **Short Index Variance**

- Payoff:
  \[ \sigma_{\text{Constituent}}^2 - \sigma_{\text{Index}}^2 = \sigma_{\text{Constituent}}^2 \times [1 - \hat{\rho}] \geq 0 \]

- Cost:
  \[ \sigma_{\text{Constituent}}^{*2} - \sigma_{\text{Index}}^{*2} = \sigma_{\text{Constituent}}^{*2} \times [1 - \hat{\rho}^*] \geq 0 \]

- Exposure: long volatility, short correlation
By underweighting the constituents’ leg with a factor $\beta = \rho^* < 1$, several benefits are obtained:

- **Vega-Neutrality**
  On trade date, if constituent variance goes up 1 point and implied correlation is unchanged, index variance would go up by $\rho^*$ points and the P&L is: $\beta \times 1\text{pt} - \rho^* \text{pts} = 0$

- **Zero cost**
  
  $\text{Cost} = \beta \sigma_{\text{Constituent}}^2 - \sigma_{\text{Index}}^2 = 0$

- **Straightforward p&l decomposition**
  
  $\text{P & L} = \text{Payoff} - \text{Cost} = \sigma_{\text{Constituent}}^2 \times [\hat{\rho}^* - \hat{\rho}]$

  Zero
2. Toy Model for Derivatives on Realised Variance
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2.1 Realised Variance: A Tradable Asset
2.2 Toy Model for Realised Variance
2.3 Application: Volatility Swap
2.4 Parameter Estimation
2.5 Model Limitations
**Variance Swap**
At expiry two parties exchange the realised variance of e.g. DJ EuroStoxx 50 daily log-returns, against a strike (‘implied variance’)

- OTC market has become very liquid on S&P 500 and DJ EuroStoxx 50, with bid-offer spreads sometimes as tight as ¼ vega.
2. Toy Model for Derivatives on Realised Variance

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2.5 Model Limitations
Fischer Black:
‘I start with the view that nothing is really constant. Volatilities themselves are not constant, and we can’t write down the process by which the volatilities change with any assurance that the process itself will stay fixed. We’ll have to keep updating our description of the process.’

Popular models (in particular Heston) for volatility or variance focus on the \textit{instantaneous, non-tradable volatility}.

Other approaches (Dupire, Buehler) focus on the \textit{variance swap curve}, which is tradable; or a \textit{fixed-term variance asset} (Duanmu, Carr-Sun).

\textbf{Toy Model}

Straightforward modification of Black-Scholes where the volatility of the variance asset $v_t$ linearly collapses as we approach its expiry $T$:

\[
\frac{d v_t}{v_t} = 2 \sqrt{\frac{T - t}{T}} d Z_t
\]
Toy Model for Realised Variance

- $v_T$ is the price of the variance asset at expiry and coincides with realised variance over the interval $[0, T]$

- $v_0$ is the fair price of the variance asset which can be observed on the variance swap market or calculated through the replicating portfolio of puts and calls

- $v_0 = \mathbb{E}(v_T)$

- The terminal distribution of $v_T$ is lognormal, making closed-form formulas for European derivatives on realised variance easy to derive
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Application: Volatility Swap

- Payoff = $\sqrt{v_T} - K_{vol}$
- With the Toy Model we find:

$$K_{vol} = \sqrt{v_0} \exp\left( -\frac{1}{6} \omega^2 T \right)$$

- Numerical example: $v_0 = 20^2 = 400$, $T = 1$, $\omega = 50\%$
  $\rightarrow K_{vol} \approx 19.2$
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**Implied approach**
Knowing $K_{vol}$ and $K_{var} (= v_0)$, we can back out an implied volatility of volatility parameter:

$$\hat{\omega}_{Implied} = \sqrt{\frac{6}{T} \ln \frac{K_{var}}{K_{vol}}}$$

**Numerical example (DAX):**
- $T = 1$
- $K_{var} \approx VDAX\ New = 17.75$
- $K_{vol} \approx ATM\ Vol = 17$

$$\omega = [6 \times \ln(17.75/17)]^{1/2} = 50.9\%$$
Historical approaches

- **Classical**: e.g. reconstitute historical time series of fixed-maturity variance prices \((v_t)_{0 \leq t \leq T}\), on a rolling basis (computationally intensive)

- **Break-even historical analysis**: e.g. find the quadratic adjustment which, on average, neutralises the P&L of an arbitrageur trading the spread between variance and volatility swaps
If volatility and variance swaps had the same strike, there would be an arbitrage:

Thus $K_{\text{vol}} < K_{\text{var}}$. Consider an arbitrageur who executes on dates $m = 1, 2, \ldots, M$ a series of normalised spread trades: BUY $1/(2K_m^2)$ units of variance at $K_m$ and SELL $(1/K_m)$ units of volatility at $K_m/\gamma$:

$$p/l = \sum_{m=1}^{M} \left[ \left( \frac{R_m^2 - K_m^2}{2K_m^2} \right) - \left( \frac{R_m - (K_m/\gamma)}{K_m} \right) \right]$$

where $R_m$ denotes realised volatility between dates $m$ and $m + \tau$. 
Parameter Estimation: Break-Even Analysis

- Assuming \( p/l = 0 \) and solving for \( \gamma \), we find:

\[
\gamma = \left[ 1 - \frac{1}{2M} \sum_{m=1}^{M} \left( \frac{R_m - K_m}{K_m} \right)^2 \right]^{-1} \equiv \hat{\Gamma}
\]

- This is the **break-even quadratic adjustment**. The corresponding theoretical volatility of volatility parameter is then given as:

\[
\hat{\omega}_{\text{Implied}} = \sqrt{\frac{6}{T} \ln \hat{\Gamma}}
\]
Toy Model for Derivatives on Realised Variance

Parameter Estimation: Break-Even Analysis

Results for the Dow Jones Euro Stoxx 50 index, using monthly trading dates $m$ between 2000 and 2005

- Index break-even quadratic adjustment (lhs)
- Index theoretical vol of vol (rhs)

Maturity (months)
2. Toy Model for Derivatives on Realised Variance

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2.5 Model Limitations
The usual Black-Scholes limitations apply: constant volatility of volatility, no transaction costs, continuous hedging.

**Specific limitations:**

- Log-normal assumption inconsistent with additivity of variance: the toy model is not suitable to model the variance swap curve, even with a time-dependent $\omega$.
- No joint dynamics with the asset price process: the toy model does not explain/take into account the equity skew.
- Consistency with vanilla option prices not considered.
3. Rational Pricing of Correlation Swaps
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3.1 Correlation Swaps
3.2 Fair Value
3.3 Parameter Estimation
3.4 Dynamic Hedging Strategy
3.5 Model Limitations
Correlation Swaps

- **Correlation Swap**
  At maturity two parties exchange the *average pair-wise realised correlation* between e.g. the DJ EuroStoxx 50 constituents, against a strike.

- OTC market, not very liquid. Introduced in early 2000’s as a means for equity exotic desks to recycle their correlation parametric risk.

- Typically *correlation swaps trade at a strike which is 5 to 15 points below implied correlation*. 
Correlation Swaps

- Correlation Swap Payoff:

\[ Payoff \equiv \frac{2}{N(N - 1)} \sum_{i<j} \rho_{i,j} = \rho_{\text{Pairwise}} \]

- The **pricing** and **dynamic hedging** of this payoff is non-trivial. However we can simplify the problem using the Proxy formulas:

\[ Payoff \approx \rho_{\text{Canonical}} \approx \hat{\rho} = \frac{\sigma^2_{\text{Index}}}{\bar{\sigma}^2_{\text{ Constituent}}} \]

which is the ratio of **two tradable assets**: index variance and average constituent variance
Rational Pricing of Correlation Swaps

How Good Is The Proxy?

- 1-month realised correlation

![Graph showing 1-month realised correlation with three lines: Realised Correl Proxy, Canonical Correl, Average Pairwise Correl (Weighted). The graph spans from January 2000 to January 2007, with values ranging from 0% to 80%.](image)
Rational Pricing of Correlation Swaps

How Good Is The Proxy?

- 24-month realised correlation

![Graph showing realised correlation proxy, canonical correlation, and average pairwise correlation (weighted). The graph plots correlation percentages over time from January 2000 to January 2007.](image-url)
3. Rational Pricing of Correlation Swaps

3.1 Correlation Swaps

3.2 Fair Value

3.3 Parameter Estimation

3.4 Dynamic Hedging Strategy

3.5 Model Limitations
Define $v_t^I$ as the **index variance** asset, $v_t^S$ as the **average constituent variance** asset, with the following forward-neutral dynamics:

\[
\frac{dv_t^I}{v_t^I} = 2\bar{\omega}_I \frac{T - t}{T} dW_t
\]

\[
\frac{dv_t^S}{\bar{v}_t^S} = 2\bar{\omega}_S \frac{T - t}{T} \left[ \chi dW_t + \sqrt{1 - \chi^2} dZ_t \right]
\]

- **Volatility of index volatility**
- **Volatility of constituent volatility**
- **Correlation between index and constituent vols**

Define $c_T \equiv \frac{v_T^I}{\bar{v}_T^S} = \hat{\rho}$ as the payoff to replicate.
Rational Pricing of Correlation Swaps

Fair value

- After calculations we find the **fair value of the correlation proxy** $\hat{\rho}$:

$$c_0 = E(c_T) = \frac{v_0^I}{v_0^I} \exp \left[ \frac{4}{3} \left( \overline{\omega}_S^2 - \overline{\omega}_S \omega_1 \chi \right) T \right]$$

- The **implied-to-fair correlation adjustment factor** is given as:

$$\frac{\hat{\rho}_0^*}{c_0} = \exp \left[ \frac{4}{3} \left( \overline{\omega}_S \omega_1 \chi - \overline{\omega}_S^2 \right) T \right]$$

- Note: For the adjustment factor to be above 1 (i.e. correlation swap strike below implied correlation, as observed on OTC markets), the correlation between index and constituent volatilities must be $>> 0$.
3. Rational Pricing of Correlation Swaps

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3.4 Dynamic Hedging Strategy
3.5 Model Limitations
Parameter Estimation: Break-Even Analysis


- Constituent break-even quadratic adjustment (lhs)
- Constituent theoretical vol of vol (rhs)
Adjustment factor for various correlation of volatilities $\chi$:

<table>
<thead>
<tr>
<th>Mat.</th>
<th>Index volatility of volatility $\omega_i$</th>
<th>Constituent volatility of volatility $\bar{\omega}_S$</th>
<th>Adjust. Factor ($\chi = 0.6$)</th>
<th>Adjust. Factor ($\chi = 0.7$)</th>
<th>Adjust. Factor ($\chi = 0.8$)</th>
<th>Adjust. Factor ($\chi = 0.9$)</th>
<th>Adjust. Factor ($\chi = 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1m</td>
<td>144.7%</td>
<td>123.4%</td>
<td>0.951</td>
<td>0.970</td>
<td>0.990</td>
<td>1.009</td>
<td>1.030</td>
</tr>
<tr>
<td>2m</td>
<td>122.6%</td>
<td>101.2%</td>
<td>0.940</td>
<td>0.966</td>
<td>0.993</td>
<td>1.021</td>
<td>1.049</td>
</tr>
<tr>
<td>3m</td>
<td>109.2%</td>
<td>88.9%</td>
<td>0.933</td>
<td>0.964</td>
<td>0.995</td>
<td>1.028</td>
<td>1.062</td>
</tr>
<tr>
<td>6m</td>
<td>86.5%</td>
<td>69.9%</td>
<td>0.920</td>
<td>0.957</td>
<td>0.997</td>
<td>1.038</td>
<td>1.081</td>
</tr>
<tr>
<td>12m</td>
<td>60.5%</td>
<td>54.1%</td>
<td>0.880</td>
<td>0.919</td>
<td>0.960</td>
<td>1.003</td>
<td>1.047</td>
</tr>
<tr>
<td>24m</td>
<td>41.5%</td>
<td>38.6%</td>
<td>0.869</td>
<td>0.906</td>
<td>0.946</td>
<td>0.987</td>
<td>1.031</td>
</tr>
</tbody>
</table>
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Rational Pricing of Correlation Swaps

Dynamic Hedging Strategy

- Hedging coefficients (deltas):
  \[ \Delta^I_t = \frac{C_t}{\nu^I_t} \]
  \[ \Delta^S_t = -\frac{C_t}{\nu^S_t} \]

- Hedging portfolio:
  \[ \Pi_t = \Delta^I_t . \nu^I_t + \Delta^S_t . \nu^S_t = \frac{C_t}{\nu^I_t} \nu^I_t - \frac{C_t}{\nu^S_t} \nu^S_t = 0 \]

  **Short vega-neutral variance dispersion**
  [Weight ratio between the constituent and index legs is equal to ‘correlation’]
3. Rational Pricing of Correlation Swaps

3.1 Correlation Swaps

3.2 Fair Value

3.3 Parameter Estimation

3.4 Dynamic Hedging Strategy

3.5 Model Limitations
In addition to the limitations of the one-factor Toy Model, the **two-factor Toy Model is not entirely arbitrage-free** as a result of the unconstrained evolution of index and constituent variance price processes:

- **The two-factor Toy Model allows for** $v_t^I > v_t^S$ !

- Also the two-factor Toy Model relies on the assumption that **constituent stocks and their weights are static**, which is only reasonable for short maturities.
Model limitations

- Model probability of terminal realised correlation $c_T > 1$, for an initial implied correlation of 50%, *ad hoc* implied volatility of volatility parameters $\omega$, and various correlation of volatilities $\chi$:

![Graph showing model probability over maturity for different $\chi$ values.](image)
Conclusion

- A correlation swap on an equity index can be quasi-replicated by dynamically trading vega-neutral variance dispersions at zero cost.
- Using a straightforward extension of Black-Scholes, we find that the fair strike of a correlation swap is equal to Implied Correlation multiplied by an adjustment factor which depends on volatility of index volatility, volatility of constituent volatility and correlation between index and constituent volatilities.
- Using a parameter estimation methodology which relies on few historical observables, we obtain numerical results supporting the intuitive idea that the adjustment factor should be close to 1.
Further research

- **Fundamental**
  Toy Model needs to be made entirely arbitrage-free.

- **Practical**
  - Fair value of other correlation measures (e.g. canonical or average pair-wise measures)
  - Free-float weights, changes in index composition

- **Numerical**
  More sophisticated parameter estimations, over longer historical periods and in other markets
References & Bibliography
References & Bibliography


- Self-referencing (available at [math.uchicago.edu/~sbossu](http://math.uchicago.edu/~sbossu))